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Front-Form Hamiltonian, Path Integral, and BRST Formulations of the Siegel Action

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The Siegel action describing the chiral bosons in one-space one-time dimension is considered on the light-front. The front-form theory is seen to possess a set of three first-class constraints and consequently a local vector gauge symmetry. The front-form Hamiltonian, path integral and BRST formulations of this front-form theory are investigated under some specific gauge choices.

KEY WORDS: front-form Hamiltonian; path integral; BRST formulations; Siegel action.

1. INTRODUCTION

The self-dual fields in one-space one-time dimensions called chiral bosons are of wide interest (Belluchi *et al.*, 1989; Bernstein and Sonnenschein, 1988; Floreanini and Jackiw, 1987; Gross *et al.*, 1985; Hanneaux and Teitelboim, 1989; Imbimbo and Schwimmer, 1987; Kulshreshtha *et al.*, 1993 Kulshershrtha and Mueller-Kirsten, 1992; Labastida and Pernici, 1987, 1988; Marcus and Schwarz, 1982; McCabe, 1989, 1990; McCabe and Mehamid, 1990; Mezinescu and Nepomechie, 1988; Siegel, 1984; Srivastava, 1989; Sonnenschein, 1988; Mstone, 1989, 1990, 1991; Wen, 1990). They are basic ingredients of some string theories (Gross *et al.*, 1985; Marcus and Schwarz, 1982) and are also important in the studies of quantum Hall effect (Stone, 1989, 1990, 1991; Wen, 1990a,b;). These fields describing chiral bosons satisfy the self-duality conditions $\partial_- \phi \equiv (\partial_0 \phi - \partial_1 \phi) = 0$. A classical covariant model describing a chiral scalar has seen proposed by Siegel (1984). Modifications of the Siegel action achieved by the addition of some appropriate Wess–Zumino terms to the action have been considered in the literature (Imbimbo and Schwimmer, 1987; Labastida and Pernici, 1987, 1988). The Becchi–Rouet–Stora and Tyutin (BRST) quantization (Becchi *et al.*, 1974; Kulshreshtha,

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1998, 2001; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993b,c,d, 1995; Nemeschansky *et al.*, 1988; Tyutin, 1975) of the Siegel action modified by the inclusion of an extra Liouville term to the original action has been investigated (Labastida and Pernici, 1987, 1988). The Hamiltonian formulation (Dirac, 1950, 1964) of the Siegel action without any modifications has been studied in Kulshreshtha *et al.* (1993a) under various gauge-fixing conditions in the instant-form (IF), and its IF-BRST formulation has been studied in Kulshreshtha *et al.*, (1999). In the present work, we propose to investigate the canonical structure, constrained dynamics, and Hamiltonian (Labastida and Pernici, 1987, 1988), path integral, and BRST (Becchi *et al.*, 1974; Kulshreshtha, 1998, 2001; Kulshreshtha and Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a,b,c,d, 1994a,b, 1995 Nemeschansky *et al.*, 1988; Tyutin, 1975) formulations of this model on the light-front (LF), i.e., on the hyperplanes: light-cone (LC) time $x^+ \equiv t = x^+ = (x^0 + x^1)/\sqrt{2} =$ constant (Dirac, 1949; Brodsky *et al.*, 1998). The Hamiltonian and BRST formulations of this model in the usual IF of dynamics (on the hyperplanes $x^0 =$ constant) (Dirac, 1949; for a recent review see e.g Brodsky *et al.*, 1998) has been investigated in (Kulshreshtha *et al.*, 1999).

The IF theory is well known to be a gauge-invariant (GI) theory possessing a set of two first-class constraints (Kulshreshtha *et al.*, 1993a). The front-form (FF) theory under the present investigation is seen to possess a set of three first-class constraints, and consequently it also describes a GI theory. The FF Hamiltonian and path integral formulation of this model has been investigated in the present work under some specific gauges.

Also, because the LF coordinates are not related to the conventional IF coordinates by a finite Lorentz transformation, the descriptions of the same physical result may be different in the IF and the FF. In fact, the quantization of relativistic field theories at fixed LC time proposed by Dirac (1949; for a recent review see e.g. Brodsky *et al.*, 1998) has very important applications and the LF variables are very useful not only in field theories but also in the description of string theories and D-brane physics. In the LC quantization (LCQ) of gauge theories the transverse degrees of freedom of the gauge field can be immediately identified as the dynamical degrees of freedom; as a result, the LCQ remains very economical in displaying the relevant degrees of freedom leading directly to the physical Hilbert space. In the context of LCQ of two-dimensional field theories, it is very often found that a theory that is gauge anomalous in the IF is no longer gauge anomalous (and therefore gauge-invariant) in the FF/LCQ. Also, in the LCQ, there is usually no conflict with the microcausality, which is in contrast with the usual IF quantization. Also, the FF has seven kinematical Poincare generators including the Lorentz boost transformations compared to only six in the usual IF framework. The advantages of the FF/LCQ over that of the conventional IF quantization are best illustrated in a recent review (Brodsky *et al.*, 1998).

However, in the usual Hamiltonian formulation of a GI theory under some gauge-fixing conditions, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under gauge-fixing). To achieve the quantization of a GI theory such that the gauge invariance of the theory is maintained even under gauge-fixing, one goes to a more generalized procedure called the BRST formulation. In the BRST formulation of a GI theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called the BRST symmetry. For this, one enlarges the Hilbert space of the GI theory and replaces the notion of the gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes the operators having different statistics. In view of this, one introduces new anticommuting variables c and $\bar{c}$ called the Faddeev–Popov ghost and antighost fields, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable b called the Nakanishi–Lautrup field (Dirac, 1950, 1964; Kulshreshtha, 1998, 2001; Kulshreshtha *et al.*, 1993b,c,d, 1994a,b, 1995; Kulshreshtha and Kulshreshtha, 1998 Nemeschansky *et al.*, 1988).

In the BRST formulation of a theory one thus embeds a GI theory into a BRST-invariant system, and the quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator Q as well as with the anti-BRST charge operator $\bar{Q}$. The new symmetry of the system (the BRST symmetry) that replaces the gauge invariance is maintained (even under gauge-fixing) and hence projecting any state onto the sector of BRST and anti-BRST invariant states yields a theory that is isomorphic to the original GI theory. The unitarity and consistency of the BRST-invariant theory described by the gauge-fixed quantum Lagrangian is guaranteed by the conservation and nilpotency of the BRST charge Q.

In the next section, we briefly consider the basics of the Siegel action in the IF of dynamics (Kulshreshtha *et al.*, 1993a, 1999). In Section 3, we study the Hamiltonian and path integral formulations of this model on the LF under gauge-fixing and in Section 4, its BRST formulation under some specific LC gauges. The summary and discussions are finally given in Section 5.

2. THE INSTANT-FORM (IF) THEORY

The Siegel action describing the chiral borons in one-space one-time dimension in the usual IF (i.e., on the hyperplanes x^0 = constant) is defined by the action (Siegel, 1984)

$$S = \int \mathscr{L} dx\, dt \tag{2.1a}$$

$$\mathscr{L} = \left[\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_1\phi)^2 + \lambda(\partial_0\phi - \partial_1\phi)\right] \tag{2.1b}$$

$$g^{\mu\nu} := \text{diag}(+1, -1) \tag{2.1c}$$

The overdots and primes here denote the time and space derivatives respectively. In the above equation, the first term corresponds to a massless boson (which is equivalent to a massless fermion), and the second term is the usual term involving the chiral-constraint $\partial_{-\phi} \equiv (\partial - 0\phi - \partial_1\phi) \approx 0$) and the auxiliary field λ. This model is seen to possess one primary constraint

$$\rho_1 = p_\lambda \approx 0 \tag{2.2}$$

and one secondary constraint

$$\rho_2 = [\pi - \partial_1\phi]/(1 + 2\lambda)^2 \approx 0 \tag{2.3}$$

which is classically equivalent to (Kulshreshtha *et al.*, 1993, 1999)

$$\rho_2 = [\pi - \partial_1\phi] \approx 0 \tag{2.4}$$

Here π and p_λ are the momenta canonically conjugate respectively to ϕ and λ. This theory is seen to possess the well-known Siegel gauge symmetry and its Hamiltonian and BRST formulations have been studied in Kulshreshtha *et al.*, (1993, 1999) under some specific gauge choices.

3. THE LIGHT-FRONT THEORY

In order to study the theory on the LF (i.e., on the hyperplanes $x^+ = (x^0 + x^1)/\sqrt{2} = \text{constant}$) one defines the LC coordinates $x^\pm := [(x^0 \pm x^1)/\sqrt{2}]$ and then writes all the quantities involved in the action in terms of $x^\pm$ instead of x^0 and x^1 (Dirac, 1949; for a recent review see e.g. Brodsky *et al.*, 1998). The action of the theory on the LF thus reads

$$S = \int \mathscr{L} dx^+ dx^- \tag{3.1a}$$

$$\mathscr{L} = [(\partial_+\phi)(\phi_-\phi + 2\lambda(\partial_-\phi)(\partial_{-\phi})] \tag{3.1b}$$

$$\partial_\pm\phi = (\partial_0\phi \pm \partial_1\phi)/\sqrt{2} \tag{3.1c}$$

As before, in (3.1b), the first term corresponds to a massless boson (which is equivalent to a massless fermion), and the second term is the usual term involving the chiral constraint $[\partial_-\phi \approx 0]$ and the auxiliary field λ. The Euler–Lagrange equations obtained from $\mathscr{L}$(3.1) are

$$[\partial_+\partial_-\phi + 2\lambda\partial_-\partial_-\phi] = 0 \tag{3.2a}$$

$$[2(\partial_-\phi)(\partial_-\phi)] = 0 \tag{3.2b}$$

3.1. The Hamiltonian and Path Integral Formulations

The LC canonical momenta obtained from $\mathscr{L}$(3.1) are

$$\pi := \frac{\partial \mathscr{L}}{\partial(\partial_+\phi)} = [\partial_-\phi] \tag{3.3a}$$

$$p_\lambda := \frac{\partial \mathscr{L}}{\partial(\partial_+\lambda)} = 0 \tag{3.3b}$$

Here, π and p_λ are the momenta canonically conjugate respectively to ϕ and λ. Also the above equations imply that the theory possesses two primary constraints:

$$\chi_1 = p_\lambda \approx 0 \tag{3.4a}$$

$$\chi_2 = [\pi - \partial_-\phi] \approx 0 \tag{3.4b}$$

The canonical Hamiltonian density corresponding to $\mathscr{L}$ is

$$\mathscr{H}_\mathrm{C} = [\pi(\partial_+\phi) + p_\lambda(\partial_+\lambda) - \mathscr{L}] = [-2\lambda(\partial_-\phi)(\partial_-\phi)] \tag{3.5}$$

After including the primary constraints χ_1 and χ_2 in the canonical Hamiltonian density $\mathscr{H}_\mathrm{C}$ with the help of Lagrange multipliers u and v, one can write the total Hamiltonian density $\mathscr{H}_\mathrm{T}$ as:

$$\mathscr{H}_\mathrm{T} = [-2\lambda(\partial_-\phi)(\partial_-\phi) + p_\lambda u + (\pi - \partial_-\phi)v] \tag{3.6}$$

The Hamiltons equations obtained from the total Hamiltonian $\mathrm{H}_\mathrm{T} = \int \mathscr{H}_\mathrm{T}\, dx^-$ are

$$\partial_+\phi = \frac{\partial H_\mathrm{T}}{\partial \pi} = v \tag{3.7a}$$

$$-\partial_+\pi = \frac{\partial H_\mathrm{T}}{\partial \phi} = [4\lambda\partial_-\partial_-\phi + \partial_- v] \tag{3.7b}$$

$$\partial_+\lambda = \frac{\partial H_\mathrm{T}}{\partial p_\lambda} = u \tag{3.7c}$$

$$-\partial_+ p_\lambda = \frac{\partial H_\mathrm{T}}{\partial \lambda} = [-2(\partial_-\phi)(\partial_-\phi)] \tag{3.7d}$$

$$\partial_+ u = \frac{\partial H_\mathrm{T}}{\partial \Pi_u} = 0 \tag{3.7e}$$

$$-\partial_+\Pi_u = \frac{\partial H_\mathrm{T}}{\partial u} = p_\lambda \tag{3.7f}$$

$$\partial_+ v = \frac{\partial H_\mathrm{T}}{\partial \pi_v} = 0 \tag{3.7g}$$

$$-\partial_+\Pi_v = \frac{\partial H_\mathrm{T}}{\partial v} = [\pi - \partial_-\phi] \tag{3.7h}$$

These are the equations of motion that preserve the constraints of the theory χ_1 and χ_2 in the course of time. For the equal LC time ($x^+ = y^+$) Poisson bracket $\{ , \}_{\rm p}$ of two functions A and B, we choose the convention

$$\{A(x), B(y)\}_{\rm p} := \int dz^- \sum_\alpha \left[\frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)} \right] \tag{3.8}$$

demanding that primary constraint χ_1 be preserved in the course of time. We obtain the secondary constraint as

$$\chi_3 := \{\chi_1, \mathscr{H}_{\rm T}\}_{\rm p} = [2(\partial_- \phi)(\partial_- \phi)] \approx 0$$

which is classically equivalent to

$$\chi_3 = [\sqrt{2}(\partial_- \phi)] \approx 0 \tag{3.9}$$

Now onwards we will consider χ_3 as the secondary Gauss law constraint of our theory (instead of $\tilde{\chi}_3$). Now the preservation of χ_2 and χ_3 for all time does not give rise to any further constraints. The theory is thus seen to possess a set of three constraints $\chi_i (i = 1, 2, 3)$:

$$\chi_1 = p_\lambda \approx 0 \tag{3.10a}$$

$$\chi_2 = [\pi - \partial_- \phi] \approx 0 \tag{3.10b}$$

$$\chi_3 = [\sqrt{2}(\partial_- \phi)] \approx 0 \tag{3.10c}$$

The matrix of the Poisson brackets of the constraints χ_i, namely, $S_{\alpha\beta}(w^-, z^-) := \{\chi_\alpha(w^-), \chi_\beta(z^-)\}_{\rm p}$, is then calculated. The nonvanishing matrix elements of the matrix $S_{\alpha\beta}(w^-, z^-)$ (with the arguments of the field variables being suppressed) are

$$S_{22} = [-2\partial_- \delta(w^- - z^-)] \tag{3.11a}$$

$$S_{23} = S_{32} = [\sqrt{2}\partial_- \delta(w^- - z^-)] \tag{3.11b}$$

The inverse of the matrix $S_{\alpha\beta}$ does not exist and therefore the matrix is singular, implying that the set of constraints χ_i is first class and that the theory is a GI theory (Belluchi *et al.*, 1989; Floreanini and Jackiw, 1987; Sonnenschein, 1988). In fact, the action of theory is seen to be invariant under the local vector gauge transformation (LVGT):

$$\delta\phi = \sqrt{2}\beta(\partial_- \phi), \quad \delta p_\lambda = 0, \quad \delta\Pi_u = 0, \quad \delta\Pi_v = 0 \tag{3.12a}$$

$$\delta\lambda = [-(\partial_+ \beta) + 2\beta(\partial_- \lambda) - \lambda(\partial_- \beta)]/\sqrt{2} \tag{3.12b}$$

$$\delta\pi = \sqrt{2}[\beta(\partial_- \partial_- \phi) + (\partial_- \beta)(\partial_- \phi)] \tag{3.12c}$$

$$\delta u = [-(\partial_+ \partial_+ \beta) + 2\beta(\partial_+ \partial_- \lambda) + 2(\partial_+ \beta)(\partial_- \lambda) - \lambda(\partial_+ \partial_- \beta)$$

$$-(\partial_+\lambda)(\partial_-\beta)]/\sqrt{2} \tag{3.12d}$$

$$\delta v = \sqrt{2}[\beta(\partial_+\partial_-\phi) + (\partial_+\beta)(\partial_-\phi)] \tag{3.12e}$$

where $\beta \equiv \beta(x^-, x^+)$ is an arbitrary function of its arguments.

The generator of the above LVGT is the charge operator of the theory:

$$J^+ = \int j^+ \, dx^- = \int dx^- \sqrt{2}[\beta(\partial_-\phi)(\partial_-\phi)] \tag{3.13}$$

The current operator of the theory is

$$J^- = \int j^- \, dx^- = \int dx^- \sqrt{2}[\beta(\partial_-\phi)\partial_+\phi + 4\lambda\partial_-\phi] \tag{3.14}$$

The divergence of the vector-current density, namely, $\partial_\mu j^\mu (= \partial_+ j^+ + \partial_- j^-)$, is therefore seen to vanish. This implies that the theory possesses at the classical level, a local vector gauge symmetry. We now proceed to quantize the theory under the gauge

$$\mathcal{G} = \lambda = 0 \tag{3.15}$$

Under this gauge, the total set of constraints of the theory becomes

$$\psi_1 = \chi_1 = p_\lambda \approx 0 \tag{3.16a}$$

$$\psi_2 = \chi_2 = [\pi - \partial_-\phi] \approx 0 \tag{3.16b}$$

$$\psi_3 = \chi_3 = [\sqrt{2}(\partial_-\phi)] \approx 0 \tag{3.16c}$$

$$\psi_4 = \mathcal{G} = \lambda = 0 \tag{3.16d}$$

The matrix of the Poisson brackets of the constraints ψ_1, namely, $T_{\alpha\beta}(w, z) := \{\psi_\alpha(w), \psi_\beta(z)\}_{\text{p}}$, is then calculated. The nonvanishing matrix elements of the matrix $T_{\alpha\beta}(w, z)$ (with the arguments of the field variables being suppressed again) are

$$T_{14} = -T_{41} = -\delta(w^- - z^-) \tag{3.17a}$$

$$T_{22} = -2\partial_-\delta(w^- - z^-) \tag{3.17b}$$

$$T_{23} = T_{32} = \sqrt{2}\partial_-\delta(w^- - z^-) \tag{3.17c}$$

The inverse of the matrix $T_{\alpha\beta}$ exists and the matrix is nonsingular. The nonvanishing elements of the inverse of the matrix $T_{\alpha\beta}$ (i.e. the elements of the matrix $(T^{-1})_{\alpha\beta}$ (with the arguments of the field variables being suppressed once again) are

$$(T^{-1})_{14} = -(T^{-1})_{41} = \delta(w^- - z^-) \tag{3.18a}$$

$$(T^{-1})_{23} = +(T^{-1})_{32} = \in(w^- - z^-)/(2\sqrt{2}) \tag{3.18b}$$

$$(T^{-1})_{33} = \in(w^- - z^-)/2 \tag{3.18c}$$

with

$$\int dz^- \, T(x^-, z^-) T^{-1}(z^-, y^-) = 1_{4\times 4} \delta(x^- - y^-) \tag{3.19}$$

with,

$$[\|\det(T_{\alpha\beta)}\|]^{1/2} = [\sqrt{2}\partial_- \delta(w^- - z^-)] \tag{3.20}$$

Now following the Dirac quantization procedure in the Hamiltonian formulation, one finds that there do not exist any nonvanishing equal LC time commutators for this theory under the gauge $\lambda = 0$. The same is seen to hold true for the quantization of the theory under some other gauge-fixing conditions such as $(\lambda - \phi) = 0$, $(\lambda - \pi) = 0$, and $(\lambda - \phi - \pi) = 0$. This is an interesting result to be noted here and its consequences need to be studied further involving the methods of constraint quantization. The path integral quantization of this theory is, however, possible as usual under all the above gauge-fixing conditions. In the following, we illustrate the path integral quantization of this theory under the gauge $\lambda = 0$, as an example.

Also, for later use (in the next section), for considering the BRST formulation of our GI theory, we convert the total Hamiltonian density $\mathscr{H}_T$ into the first-order Lagrangian density

$$\mathscr{L} = [\pi(\partial_+\phi) + p_\lambda(\partial_+\lambda) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) - \mathscr{H}_T] \tag{3.21a}$$

$$= [2\lambda(\partial_-\phi)(\partial_-\phi) + (\partial_-\phi)(\partial_+\phi) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v)] \tag{3.21b}$$

In the above equation the terms $p_\lambda(\partial_+\lambda) - u$ and $\pi(\partial_+\phi - v)$ drop out in view of the Hamiltons equations of the theory.

The transition to quantum theory in the path integral formulation is made by writing the vacuum-to-vacuum transition amplitude called the generating functional $Z[J_i]$ in the presence of external source currents J_i under the gauge $\zeta = \lambda \approx 0$ as (see e.g., Gitman and Tyutin, 1990, Henneaux and Teitelboim, 1992)

$$Z[J_i] = \int [d\mu] \exp\left[i \int dx^+ dx^- \right] [J_i \phi^i + 2\lambda(\partial_-\phi)(\partial_-\phi) + (\partial_-\phi)(\partial_+\phi) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v)] \tag{3.22a}$$

where ϕ^i are the phase space variables

$$\phi^i \equiv (\phi, \lambda, u, v) \tag{3.22b}$$

and the functional measure $[d_\mu]$ for the above generating functional is

$$[d_\mu] = [\sqrt{2}\partial_-\delta(w^- - z^-)][d\phi][d\pi][d\lambda][dp_\lambda][du] [d\Pi_u][dv][d\Pi_v]\delta[(p_\lambda) \approx 0]\delta[(\pi - \partial_-\phi) \approx 0] \delta[\sqrt{2}(\partial_-\phi) \approx 0]\delta[(\lambda) \approx 0] \tag{3.22c}$$

4. THE BRST FORMULATION

We now rewrite our GNLSM, which is GI as a quantum system that possesses the generalized gauge-invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our GI GNLSM and replace the notion of gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics. We then introduce new anticommuting variables c and $\bar{c}$ (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable b (called the Nakamishi–Lautrup field) such that (Becchi *et al.*, 1974; Kulshreshtha, 1998, 2001; Kulshreshtha *et al.*, 1993b,c,d, 1994a,b, 1995, 1998; Nemeschansky *et al.*, 1988; Tyutin, 1975; Kulshreshtha and Kulshreshtha, 1998)

$$\hat{\delta}\phi = \sqrt{2}c(\partial_-\phi), \quad \hat{\delta}p_\lambda = 0, \quad \hat{\delta}\Pi_u = 0, \quad \hat{\delta}\Pi_v = 0 \tag{4.1a}$$

$$\hat{\delta}\lambda = [-(\partial_+ c) + 2c(\partial_-\lambda) - \lambda(\partial_- c)]/\sqrt{2} \tag{4.1b}$$

$$\hat{\delta}\pi = \sqrt{2}[c(\partial_-\partial_-\phi) + (\partial_- c)(\partial_-\phi)] \tag{4.1c}$$

$$\hat{\delta}u = [-(\partial_+\partial_+ c) + 2c(\partial_+\partial_-\lambda) + 2(\partial_+ c)(\partial_-\lambda) - \lambda(\partial_+\partial_- c) - (\partial_+\lambda)(\partial_- c)]/\sqrt{2} \tag{4.1d}$$

$$\hat{\delta}v = \sqrt{2}[c(\partial_+\partial_-\phi) + (\partial_+ c)(\partial_-\phi)] \tag{4.1e}$$

$$\hat{\delta}c = 0, \quad \hat{\delta}\bar{c} = b, \quad \hat{\delta}b = 0 \tag{4.1f}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical variables to be a function $f(\pi, p_\lambda, \Pi_u, \Pi_v, p_b, \Pi_c, \Pi_{\bar{c}}, \pi, \lambda, u, v, b, c, \bar{c})$ such that $\hat{\delta} f = 0$.

4.1. Gauge Fixing in the BRST Formulism

Permorming gauge-fixing in the BRST formalism implies adding to the first-order Lagrangian density $\mathscr{L}_{10}$, a trivial BRST-invariant function (Becchi *et al.*, 1974; Kulshreshtha, 1998, 2001; Kulshreshtha *et al.*, 1993b,c,d, 1994a,b, 1995; Nemeschansky *et al.*, 1988; Tyutin, 1975). We thus write

$$\mathscr{L}_{\mathrm{BRST}} = \left[2\lambda(\partial_-\phi)(\partial_-\phi) + (\partial_-\phi)(\partial_+\phi) + \Pi_u(\partial_+ u) + \Pi_v(\partial_+ v) + \hat{\delta}\,\bar{c}\left(-\sqrt{2}\partial_+\lambda + \frac{1}{2}b\right)\right] \tag{4.2}$$

The last term in the above equation is the extra BRST-invariant gauge-fixing term. After one integration by parts, the above equation could now be written as

$$\mathscr{L}_{\mathrm{BRST}} = \Big[2\lambda(\partial_-\phi)(\partial_-\phi) + (\partial_-\phi)(\partial_+\phi) + \Pi_u(\partial_+u) + \Pi_v(\partial_+v) - \sqrt{2}b(\partial_+\lambda) + \frac{1}{2}b^2 + (\partial_+\overline{c})(\partial_+\overline{c})\Big] \tag{4.3}$$

Proceeding classically, the Euler–Lagrange equation for b reads

$$b = \sqrt{2}(\partial_+\lambda) \tag{4.4}$$

The requirement $\hat{\delta}b = 0$ then implies

$$\hat{\delta}b = \sqrt{2}\hat{\delta}(\partial_+\lambda) \tag{4.5}$$

which in turn implies

$$\partial_+\partial_+c = 0 \tag{4.6}$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathscr{L}_{\mathrm{BRST}}$ with respect to $\overline{c}$. In introducing momenta one has to be careful in defining those for the fermionic variables. We thus define the bosonic momenta in the usual manner so that

$$p_\lambda := \frac{\partial}{\partial(\partial_+\lambda)}\mathscr{L}_{\mathrm{BRST}} = -\sqrt{2}b \tag{4.7}$$

but for the fermionic momenta with directional derivatives we set

$$\Pi_{\mathrm{c}} = \mathscr{L}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial(\partial_+c)} = (\partial_+\overline{c}), \qquad \Pi_{\overline{\mathrm{c}}} = \frac{\overrightarrow{\partial}}{\partial(\partial_+\overline{c})}\mathscr{L}_{\mathrm{BRST}} = (\partial_+c) \tag{4.8}$$

implying that the variable canonically conjugate to c is $(\partial_+\overline{c})$ and the variable conjugate to $\overline{c}$ is (∂_+c). For writing the Hamiltonian density from the Lagrangian density in the usual manner we remember that the former has to be Hermitian so that

$$\mathscr{H}_{\mathrm{BRST}} = [\pi(\partial_+\phi) + p_\lambda(\partial_+\lambda) + \Pi_u(\partial_+u) + \Pi_v(\partial_+v) + \Pi_{\mathrm{C}}(\partial_+c) + (\partial_+\overline{c})\Pi_{\overline{c}} - \mathscr{L}_{\mathrm{BRST}}] \tag{4.9a}$$

$$= \Big[p_\lambda u + \pi v - (\partial_+\phi)(\partial_-\phi) - 2\lambda(\partial_-\phi)(\partial_-\phi) + \frac{1}{4}(p_\lambda)^2 + \Pi_{\mathrm{c}}\Pi_{\overline{c}}\Big] \tag{4.9b}$$

We can check the consistency of (4.8) and (4.9) by looking at Hamilton's equations for the fermionic variables, i.e.,

$$\partial_+ c = \frac{\vec{\partial}}{\partial \Pi_c} \mathscr{H}_{\text{BRST}}, \qquad \partial_+ \bar{c} = \mathscr{H}_{\text{BRST}} \frac{\overleftarrow{\partial}}{\partial \Pi_{\bar{c}}} \tag{4.10}$$

Thus we see that

$$\partial_+ c = \frac{\vec{\partial}}{\partial \Pi_c} \mathscr{H}_{\text{BRST}} = \Pi_{\bar{c}}, \qquad \partial_+ \bar{c} = \mathscr{H}_{\text{BRST}} \frac{\vec{\partial}}{\partial \Pi_{\bar{c}}} = \Pi_c \tag{4.11}$$

is in agreement with (4.8). For the operators c, $\bar{c}$, $\partial_+ c$ and $\partial_+ \bar{c}$, one needs to satisfy the anticommutation relations of $\partial_+ c$ with $\bar{c}$ or of $\partial_+ \bar{c}$ with c, but not of c with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that

$$\{\Pi_{\text{c}}, \Pi_{\bar{\text{c}}}\} = \{\bar{\text{c}}, c\} = 0, \qquad \partial_+ \{\bar{\text{c}}, c\} = 0 \tag{4.12a}$$

$$\{\partial_+ \bar{\text{c}}, c\} = (-1)\{\partial_+ \text{c}, c\} \tag{4.12b}$$

where {, } means an anticommutator. We thus see that the anticommulators in (4.12b) are nontrivial and need to be fixed. In order to fix these, we demand that c satisfy the Heisenberg equation (Becchi *et al.*, 1974; Kulshreshtha, 1998, 2001; Kulshreshtha *et al.*, 1993b,c,d, 1994a,b, 1995; Nemeschansky *et al.*, 1988; Tyutin, 1975):

$$[c, \mathscr{H}_{\text{BRST}}] = i \partial_+ c \tag{4.13}$$

and using the property $c^2 = \bar{c}^2 = 0$ one obtains

$$[c, \mathscr{H}_{\text{BRST}}] = (\partial_+ \bar{c}, c) \partial_+ c \tag{4.14}$$

Eqs. (4.12)–(4.14) then imply

$$\{\partial_+ \bar{\text{c}}, c\} = (-1)\{\partial_+ \text{c}, c\} = i \tag{4.15}$$

Here the minus sign in the above equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory (Becchi *et al.*, 1974; Kulshreshtha, 1998, 2001; Kulshreshtha *et al.*, 1993b,c,d, 1994a,b, 1995; Nemeschansky *et al.*, 1988; Tyutin, 1975).

4.2. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformations (4.1). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators that satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy

$$[\phi, Q] = \partial_+ c, \qquad [\pi, Q] = -[\sqrt{2}\partial_- c + \partial_-\partial_+ c],$$
$$[\lambda, Q] = \partial_+ c \tag{4.16a}$$
$$\{\bar{c}, Q\} = [\partial_-\phi - p_\lambda - \pi], \qquad \{\partial_+\bar{c}, Q\} = -\sqrt{2}(\partial_-\phi) \tag{4.16b}$$

All other commutators and anticommutators involving Q vanish. In view of (4.16), the BRST charge operator of the present theory can be written as

$$Q = \int dx^- \, [ic[\sqrt{2}\partial_-\phi] - i(\partial_+ c)[p_\lambda + \pi - \partial_-\phi] \tag{4.17}$$

This equation implies that the set of states satisfying the conditions

$$p_\lambda|\psi\rangle = 0 \tag{4.18a}$$
$$[\pi - \partial_-\phi]|\psi\rangle = 0 \tag{4.18b}$$
$$[\sqrt{2}\partial_-\phi]|\psi\rangle = 0 \tag{4.18c}$$

belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belongs to the set of BRST-invariant states.

In order to understand the condition needed for recovering the physical states of the theory we rewrite the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider (4.6). The solution of this equation (4.6) gives (for the LC time $x^+ \equiv t$) the Heisenberg operator $c(t)$ (and correspondingly $\bar{c}(t)$) as

$$c(t) = Gt + F, \qquad \bar{c}(t) = G^\dagger t + F^\dagger \tag{4.19}$$

which at LC time $t = 0$ imply

$$c \equiv c(0) = F, \qquad \bar{c} \equiv \bar{c}(0) = F^\dagger \tag{4.20a}$$
$$\partial_+ c \equiv \partial_+ c(0) = G, \qquad \partial_+\bar{c} \equiv \partial_+\bar{c}(0) = G^\dagger \tag{4.20b}$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\partial_+\bar{c}, \partial_+ c\} = 0 \tag{4.21a}$$
$$\{\partial_+\bar{c}, c\} = i = -\{\partial_+ c, \bar{c}\} \tag{4.21b}$$

we then obtain

$$F^2 = F^{\dagger 2} = \{F^\dagger, F\} = \{G^\dagger, G\} = 0 \tag{4.22a}$$
$$\{G^\dagger, F\} = i, \qquad \{G, F^\dagger\} = -i \tag{4.22b}$$

We now let $|0\rangle$ denote the fermionic vacuum for which

$$G|0\rangle = F|0\rangle = 0 \tag{4.23}$$

Defining $|0\rangle$ to have norm one, (4.22b) implies

$$\langle 0|FG^{\dagger}|0\rangle = i, \qquad \langle 0|GF^{\dagger}|0\rangle = -i \tag{4.24}$$

So that

$$G^{\dagger}|0\rangle = 0, \qquad F^{\dagger}|0\rangle = 0 \tag{4.25}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathscr{H}_{\text{BRST}}$ is however irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators

$$\mathscr{H}_{\text{BRST}} = \Big[p_{\lambda}u + \pi v - (\partial_{+}\phi)(\partial_{+}\phi) - 2\lambda(\partial_{-}\phi)(\partial_{-}\phi) + \frac{1}{4}(p_{\lambda})^{2} + G^{\dagger}G\Big] \tag{4.26}$$

and the BRST charge operator Q is

$$Q = \int dx^{-}\,[iF[\sqrt{2}\partial_{-}\phi] - iG(p_{\lambda} + \pi - \partial_{-}\phi)] \tag{4.27}$$

Now because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (4.18) hold but also additional states for which

$$B|\psi\rangle = D|\psi\rangle = 0 \tag{4.28a}$$

$$p_{\lambda}|\psi\rangle \neq 0 \tag{4.28b}$$

$$[\pi - \partial_{-}\phi]|\psi\rangle \neq 0 \tag{4.28c}$$

$$[\sqrt{2} - \partial_{-}\phi]|\psi\rangle \neq 0 \tag{4.28d}$$

The Hamiltonian is also invariant under the anti-BRST transformation given by

$$\hat{\delta}\phi = -\sqrt{2}\bar{c}(\partial_{-}\phi), \quad \hat{\delta}p_{\lambda} = 0, \quad \hat{\delta}\Pi_{u} = 0, \quad \hat{\delta}\Pi_{v} = 0 \tag{4.29a}$$

$$\hat{\delta}\lambda = [+(\partial_{+}\bar{c}) - 2\bar{c}(\partial_{-}\lambda) + \lambda(\partial_{-}\bar{c})]/\sqrt{2} \tag{4.29b}$$

$$\hat{\delta}\pi = \sqrt{2}[-\bar{c}(\partial_{-}\partial_{-}\phi) - (\partial_{-}\bar{c})(\partial_{-}\phi)] \tag{4.29c}$$

$$\hat{\delta}u = [+(\partial_{+}\partial_{+}\bar{c}) - 2\bar{c}(\partial_{+}\partial_{-}\lambda) - 2(\partial_{+}\bar{c})(\partial_{-}\lambda) + \lambda(\partial_{+}\partial_{-}\bar{c}) + (\partial_{+}\lambda)(\partial_{-}\bar{c})]/\sqrt{2} \tag{4.29d}$$

$$\hat{\delta}v = \sqrt{2}[-\bar{c}(\partial_{+}\partial_{-}\phi) - (\partial_{+}\bar{c})(\partial_{-}\phi)] \tag{4.29e}$$

$$\hat{\delta}\bar{c} = 0, \qquad \hat{\delta}c = -b, \qquad \hat{\delta}b = 0 \tag{4.29f}$$

with the generator or anti-BRST charge

$$\bar{Q} = \int dx^- \, [-i\bar{c}[\sqrt{2}\partial_-\phi] + i(\partial_+\bar{c})(p_\lambda + \pi - \partial_-\phi) \tag{4.30a}$$

$$= \int dx^- \, [-iF^\dagger[\sqrt{2}\partial_-\phi] + iG^\dagger(p_\lambda + \pi - \partial_-\phi)] \tag{4.30b}$$

We also have

$$\partial_+ Q = [Q, H_{\text{BRST}}] = 0 \tag{4.31a}$$

$$\partial_+ \bar{Q} = [\bar{Q}, H_{\text{BRST}}] = 0 \tag{4.31b}$$

with

$$H_{\text{BRST}} = \int dx \, \mathscr{H}_{\text{BRST}} \tag{4.31c}$$

and we further impose the dual condition that both Q and $\bar{Q}$ annihilate physical states, implying that

$$Q|\psi\rangle = 0 \quad \text{and} \quad \bar{Q}|\psi\rangle = 0 \tag{4.32}$$

The states for which (4.18) hold, satisfy both of these conditions and, in fact, are the only states satisfying both of these conditions, since, although with (4.22)

$$G^\dagger G = -GG^\dagger \tag{4.33}$$

there are no states of this operator with $G^\dagger|0\rangle = 0$ and $F^\dagger|0\rangle = 0$ [cf. (4.25)], and hence no free eigenstates of the fermionic part of H_{BRST} that are annihilated by each of $G, G^\dagger, F, F^\dagger$. Thus the only states satisfying (4.32) are those satisfying the constraints of the throry.

Further, the states for which (4.18) hold satisfy both the conditions (4.32) and infact, are the only states satisfying both of these conditions because in view of (4.21) one cannot have simultaneously c, $\partial_+ c$ and $\bar{c}$, $\partial_+\bar{c}$, applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (4.32) are those that satisfy the constraints of the theory and they belong to the set of BRST-invariant and anti-BRST-invariant states.

Alternatively, one can understand the above point in terms of fermionic annihilation and creation operators as follows. The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which (4.18) hold but also additional states for which (4.28) hold. However, $\bar{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\bar{Q}$ contains only the states for which (4.18) hold, simply because $G^\dagger|\psi\rangle \neq 0$ and $F^\dagger|\psi\rangle \neq 0$. Thus in this alternative way also we see that the states satisfying $Q|\psi\rangle\bar{Q}|\psi\rangle = 0$ (i.e., satisfying (4.32)) are only those states that satisfy the constraints of the theory and also that these states belong to the set of BRST invariant and anti-BRST-invariant states.

5. SUMMARY AND DISCUSSIONS

In this work we have studied the Siegel action describing the chiral bosons on the LF, i.e., on the hyperplanes $x^+ = (x^0 + x^1)/\sqrt{2}$ constant. The theory in the IF has been studied before (Kulshreshtha *et al.*, 1993a, 1999).

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